

## Adiabatic Decoherence

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We study a general quantum system interacting with environment modeled by a bosonic heat bath of Caldeira and Leggett type. General interaction Hamiltonians are considered that commute with the system's Hamiltonian so that there is no energy exchange between the system and bath. We argue that this model provides an appropriate description of adiabatic quantum decoherence, i.e., loss of entanglement on time scales short compared to those of thermal relaxation processes associated with energy exchange with the bath. The interaction Hamiltonian is then proportional to a conserved "pointer observable." Calculation of the elements of the reduced density matrix of the system is carried out exactly, and the time-dependence of decoherence is identified, similar to recent results for related models. Our key finding is that the decoherence process is controlled by spectral properties of the interaction rather than the system's Hamiltonian.

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**KEY WORDS:** Quantum decoherence, heat bath, effects of environment, dissipation.

### 1. INTRODUCTION

Quantum decoherence, dissipation, and thermalization due to interactions with environment have long been important fundamental issues theoretically and experimentally.<sup>(1-11)</sup> Decoherence and related topics have attracted much interest recently due to rapid development of new fields such as quantum computing and quantum information theory.<sup>(12-18)</sup> Decoherence due to external interactions is a major obstacle in the way of implementation of devices such as quantum computers. Thus in addition to studies of the physics of decoherence processes there emerged a new field of quantum error correction<sup>(19-25)</sup> aiming at effective stabilization of quantum states against decoherence essentially by involving many additional quantum

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systems and utilizing redundancy. The present work contributes to the former topic: the physics of decoherence.

Decoherence is a result of the coupling of the quantum system under consideration to the environment which, generally, is the rest of the universe. In various experimentally relevant situations the interaction of the quantum system with environment is dominated by the system's microscopic surroundings. For example, the dominant source of such interaction for an atom in an electromagnetic cavity is the electromagnetic field itself coupled to the dipole moment of the atom.<sup>(26)</sup> In case of Josephson junction in a magnetic flux<sup>(27)</sup> or defect propagation in solids, the interaction can be dominated by acoustic phonons or delocalized electrons.<sup>(28)</sup> Magnetic macromolecules interact with the surrounding spin environment such as nuclear spins.<sup>(18)</sup> Numerous other specific examples could be cited.

In this work we aim at a general phenomenological description that models the physically important effects of external interactions as far as adiabatic decoherence, to be defined later, is concerned. We note that generally thermalization and decoherence are associated with the interaction of the quantum system, described in isolation by the Hamiltonian  $H_S$ , with another, large system which we will term the "bath" and which internally has the Hamiltonian  $H_B$ . The actual interaction will be represented by the Hamiltonian  $H_I$  so that the total Hamiltonian of the system,  $H$ , is

$$H = H_S + H_B + H_I \quad (1.1)$$

It is important to realize that typically the bath is a large, macroscopic system. Truly irreversible interactions of a quantum system with its environment, such as thermal equilibration or decoherence associated with measurement processes, can only be obtained in the Hamiltonian description (1.1) when it is supplemented by taking the limit of the number of particles or degrees of freedom of the bath going to infinity.

Interactions of a quantum system with macroscopic systems can lead to different outcomes. For instance, interaction with a true "heat bath" leads to thermalization: the reduced density matrix of the system approaches  $\exp(-\beta H_S)$  for large times. Here

$$\beta = 1/(kT) \quad (1.2)$$

as usual, and by "reduced" we mean the density matrix traced over the states of the bath. On the other hand for decoherence we expect the reduced density matrix to approach a diagonal form in the "preferred basis" somehow selected by the "pointer observable" Hermitian operator<sup>(1-6, 29, 30)</sup> which is thereby "measured" by the macroscopic system (bath).

It is important to realize that study of decoherence in the present context does not fully resolve the problem of understanding quantum measurement and other fundamental issues at the borderline of quantum and classical behaviors, such as, for instance, the absence of macroscopic manifestations of Schrödinger-cat type quantum superposition of states. The more optimistic recent literature<sup>(4-6)</sup> considers description of entanglement and decoherence the key to such understanding. However, these fundamental problems have remained open thus far.

The most explored and probably most tractable approach to modeling the environmental interactions has involved representing their effects by coupling the original quantum system to a set of noninteracting harmonic oscillators (bosonic heat bath).<sup>(1, 2, 8-11, 14, 31-33)</sup> Fermionic heat bath can be also considered, e.g., ref. 34. We will use the term "heat bath" for such systems even when they are used for other than thermalization studies because they have the temperature parameter defined via initial conditions, as described later.

Rigorous formulation of the bosonic heat bath approach was initiated by Ford, Kac and Mazur<sup>(32)</sup> and more recently by Caldeira and Leggett.<sup>(11, 29)</sup> It has been established, for harmonic quantum systems, that the influence of the heat bath described by the oscillators is effectively identical to the external uncorrelated random force acting on a quantum system under consideration. In order for the system to satisfy equation of motion with a linear dissipation term in the classical limit the coupling was chosen to be linear in coordinates while the coupling constants entered lumped in a spectral function which was assumed to be of a power-law form in the oscillator frequency, with the appropriate Debye cutoff. We will make this concept more explicit later.

This model of a heat bath was applied to studying effects of dissipation on the probability of quantum tunneling from a metastable state.<sup>(8, 29)</sup> It was found that coupling a quantum system to the heat bath actually decreases the quantum tunneling rate. The problem of a particle in a double well potential was also considered.<sup>(9, 33)</sup> In this case the interaction with the bath leads to quantum coherence loss and complete localization at zero temperature. This study has led to the spin-boson Hamiltonian<sup>(9, 10)</sup> which found numerous other applications. The Hilbert space of the quantum systems studied was effectively restricted to the two-dimensional space corresponding to the two lowest energy levels.

Another possible application of the bosonic heat bath model concerns aspects of quantum measurement. It is believed that the bath is an intrinsic part of a measuring device. In other words, it continuously monitors the physical quantity whose operator is coupled to it.<sup>(4-6)</sup> It has been shown in the exactly solvable model of the quantum oscillator coupled to a heat

bath<sup>(5)</sup> that the reduced density matrix of the quantum system decoheres, i.e., loses its off-diagonal elements representing the quantum correlations in the system, in the eigenbasis of the interaction Hamiltonian. It has also been argued that the time scale on which this “measurement” occurs is much less than the characteristic time for thermal relaxation of the system.

It is natural to assume that if such a “bath” description of the process of measurement of a Hermitian operator  $A_S$  exists, then the interaction Hamiltonian  $H_I$  in (1.1) will involve  $A_S$  as well as some bath-Hilbert-space operators. No general description of this process exists. Furthermore, when we are limited to specific models in order to obtain tractable, e.g., analytically solvable, examples, then there is no general way to separate decoherence and thermalization effects. We note that thermalization is naturally associated with exchange of energy between the quantum system and heat bath. Model system results and general expectations mentioned earlier suggest that at least in some cases decoherence involves its own time scales which are shorter than those of approach to thermal equilibrium.

In this work we propose to study *adiabatic decoherence*, i.e., a special case of no energy exchange between the system and bath. Thus we assume that  $H_S$  is conserved, i.e.,  $[H_S, H] = 0$ . This assumption is a special case of “quantum nondemolition measurement” concept<sup>(2,30)</sup> exemplified by the Kerr effect, for instance. Since  $H_S$  and  $H_B$  in (1.1) operate in different Hilbert spaces, this is equivalent to requiring

$$[H_S, H_I] = 0 \quad (1.3)$$

Furthermore, we will assume that  $H_I$  is linear in  $A_S$ :

$$H_I = A_S P_B \quad (1.4)$$

where  $P_B$  acts in the Hilbert space of the bath. Then we have

$$[A_S, H_S] = 0 \quad (1.5)$$

Thus, we consider cases in which the measured, “pointer” observable  $A_S$  is one of the conserved quantities of the quantum system when it is isolated. Interaction with the bath will then correspond to measurement of such an observable, which can be the energy itself. Specifically, the model of ref. 14 corresponds to  $A_S = H_S$  for the case of the spin- $\frac{1}{2}$  two-state system, motivated by quantum-computing applications; see also refs. 2, 12–15. The models of refs. 1 and 2 correspond to the choices of  $A_S = H_S$  and  $A_S = f(H_S)$ , respectively, for a system coupled to a bosonic spin bath, where  $f$  is an arbitrary well-behaved function.

Here we derive exact results for adiabatic decoherence due to coupling to the bosonic heat bath, assuming general  $A_S$  that commutes with  $H_S$ . While technically this represents an extension of the results of refs. 1 and 2, we demonstrate that the general case reveals certain new aspects of the decoherence process. Our new exact-solution method utilizes coherent states and may be of interest in other applications as well. In Section 2, we define the system. Specifically, we choose the bosonic heat bath form for  $H_B$  and  $P_B$  in (1.1) and (1.4), but we keep  $H_S$  and  $A_S$  general. However, we also analyze the mechanism leading to exact solvability of general models of this type. Section 3 reports our derivation of the exact expression for the reduced density matrix of the system. Discussion of the results and definition of the continuum limit are given in Section 4.

## 2. MODELS OF ADIABATIC DECOHERENCE

We will be mainly interested in the following Hamiltonian for the quantum system coupled to a bath of bosons (harmonic oscillators) labeled by the subscript  $k$ :

$$H = H_S + \sum_k \omega_k a_k^\dagger a_k + A_S \sum_k (g_k^* a_k + g_k a_k^\dagger) \quad (2.1)$$

Here  $a_k^\dagger$  and  $a_k$  are bosonic creation and annihilation operators, respectively, so that their commutation relation is  $[a_k, a_k^\dagger] = 1$ . The second term in (2.1) represents the free field or Hamiltonian of the heat bath  $H_B$ . The last term is the interaction Hamiltonian  $H_I$ . The coupling constants will be specified later; exact results obtained in Section 3 apply for general  $\omega_k$  and  $g_k$ . Here and in the following we use the convention

$$\hbar = 1 \quad (2.2)$$

and we also assume that the energy levels of each oscillator are shifted by  $\frac{1}{2}\omega_k$  so that the ground state of each oscillator has zero energy.

Since we assume that  $H_S$  and  $A_S$  commute, we can select a common set  $|i\rangle$  of eigenstates:

$$H_S |i\rangle = E_i |i\rangle \quad (2.3)$$

$$A_S |i\rangle = \lambda_i |i\rangle \quad (2.4)$$

One of the simplifications here, due to the fact that  $H_S$  and  $A_S$  commute, is that these eigenstates automatically constitute the “preferred basis” mentioned earlier.

We will assume that initially the quantum system is in a pure or mixed state described by the density matrix  $\rho(0)$ , not entangled with the bath. For the bath, we assume that each oscillator is independently thermalized (possibly by prior contact with a “true” heat bath) at temperature  $T$ , with the density matrix  $\theta_k$ . The total system-plus-bath density matrix will then be the product

$$\rho(0) \prod_k \theta_k \quad (2.5)$$

Here

$$\theta_k = Z_k^{-1} e^{-\beta \omega_k a_k^\dagger a_k} \quad (2.6)$$

$$Z_k \equiv (1 - e^{-\beta \omega_k})^{-1} \quad (2.7)$$

where  $Z_k$  is the partition function for the oscillator  $k$ . The quantity  $\beta$  was defined in (1.2). Introduction of the temperature parameter via the initial state of the bath is common in the literature.<sup>(1, 2, 8–11, 14–17, 29, 32, 33)</sup> While it may seem artificial, we recall that the bath is supposed to be a large system presumably remaining thermalized on the time scales of interest. Specific results indicating that the bosonic heat bath can be viewed as a source of thermalizing noise have been mentioned earlier; see also ref. 35.

Our objective is to study the reduced density matrix of the system at time  $t \geq 0$ ; it has the following matrix elements in the preferred basis:

$$\rho_{mn}(t) = \text{Tr}_B \left[ \langle m | e^{-iHt} \left( \rho(0) \prod_k \theta_k \right) e^{iHt} | n \rangle \right] \quad (2.8)$$

Here the outer trace is taken over the states of the heat bath, i.e., the bosonic modes. The inner matrix element is in the space of the quantum system. Note that for no coupling to the bath, i.e., for  $g_k = 0$ , the density matrix of the system is simply

$$[\rho_{mn}(t)]_{g_k=0} = \rho_{mn}(0) e^{i(E_n - E_m)t} \quad (2.9)$$

For the interacting system, the heat-bath states must be summed over in the trace in (2.8). It is instructive to consider a more general case with the bath consisting of independent “modes” with the Hamiltonians  $M_k$ , so that

$$H_B = \sum_k M_k \quad (2.10)$$

where for the bosonic bath we have  $M_k = \omega_k a_k^\dagger a_k$ . Similarly, for the interaction term we assume coupling to each mode independently,

$$H_I = A_S \sum_k J_k \quad (2.11)$$

where for the bosonic bath we have  $J_k = g_k^* a_k + g a_k^\dagger$ . Relation (2.5) remains unchanged, with the definitions (2.6) and (2.7) replaced by

$$\theta_k = Z_k^{-1} e^{-\beta M_k} \quad (2.12)$$

$$Z_k = \text{Tr}_k [e^{-\beta M_k}] \quad (2.13)$$

where the trace is over a single mode  $k$ .

Owing to the fact that  $H_S$  and  $A_S$  share common eigenfunctions, the inner matrix element calculation in (2.8), in the system space, can be expressed in terms of the eigenvalues defined in (2.3)–(2.4). Specifically, we define the bath-space operators

$$h_i = E_i + \sum_k M_k + \lambda_i \sum_k J_k \quad (2.14)$$

which follow from the form of the Hamiltonian. The calculation in (2.8) then reduces to

$$\rho_{mn}(t) = \text{Tr}_B \left[ \langle m | e^{-ih_m t} \left( \rho(0) \prod_k \theta_k \right) e^{ih_n t} | n \rangle \right] \quad (2.15)$$

which yields the expression

$$\rho_{mn}(t) = \rho_{mn}(0) \text{Tr}_B \left[ e^{-ih_m t} \left( \prod_k \theta_k \right) e^{ih_n t} \right] \quad (2.16)$$

We will now assume that the operators of different modes  $k$  commute. This is obvious for the bosonic or spin baths and must be checked explicitly if one uses the present formulation for a fermionic bath. Then we can factor the expression (2.16) as follows:

$$\rho_{mn}(t) = \rho_{mn}(0) e^{i(E_n - E_m)t} \prod_k \left\{ \text{Tr}_k \left[ e^{-i(M_k + \lambda_m J_k)t} \theta_k e^{i(M_k + \lambda_n J_k)t} \right] \right\} \quad (2.17)$$

This expression, or variants derived in earlier works,<sup>1,2,14</sup> suggests that the problem is exactly solvable in some cases. Indeed, the inner trace is over a *single mode* of the bath. For a spin bath of spin- $\frac{1}{2}$  “modes” the calculation

involves only  $(2 \times 2)$ -matrix manipulations and is therefore straightforward.<sup>(2, 14)</sup> However, in this case the only nontrivial choice of the “pointer observable” corresponds, in our notation, to  $A_S = H_S$ , with both operators usually chosen equal to the Pauli matrix  $\sigma_z$ . There is also hope for obtaining analytical results for other baths with modes in finite-dimensional spaces, such as spins other than  $\frac{1}{2}$ ; we have not explored this possibility.

For the bosonic spin bath, the calculation is in the space of a single harmonic oscillator. It can be carried out by using operator identities.<sup>1, 2</sup> We have used instead a method based on the coherent-state formalism which is detailed in the next section.

### 3. EXACT SOLUTION FOR THE DENSITY MATRIX

We utilize the coherent-state formalism, e.g., refs. 35, 36. The coherent states  $|z\rangle$  are the eigenstates of the annihilation operator  $a$  with complex eigenvalues  $z$ . Note that from now on we omit the oscillator index  $k$  whenever this leads to no confusion. These states are not orthogonal:

$$\langle z_1 | z_2 \rangle = \exp(z_1^* z_2 - \frac{1}{2} |z_1|^2 - \frac{1}{2} |z_2|^2) \quad (3.1)$$

They form an over-complete set, and one can show that the identity operator in a single-oscillator space can be obtained as the integral

$$\int d^2z |z\rangle \langle z| = 1 \quad (3.2)$$

Here the integration by definition corresponds to

$$d^2z \equiv \frac{1}{\pi} d(\text{Re } z) d(\text{Im } z) \quad (3.3)$$

Furthermore, for an arbitrary operator  $A$ , we have, in a single-oscillator space,

$$\text{Tr } A = \int d^2z \langle z | A | z \rangle \quad (3.4)$$

Finally, we note the following identity,<sup>(35)</sup> which will be used later,

$$e^{\Omega a^\dagger a} = \mathcal{N} [e^{a^\dagger (e^\Omega - 1) a}] \quad (3.5)$$

In this relation  $\Omega$  is an arbitrary c-number, while  $\mathcal{N}$  denotes normal ordering.



The result (2.17) for the reduced density matrix, assuming the bosonic spin bath, can be written as

$$\rho_{mn}(t) = \rho_{mn}(0) e^{i(E_n - E_m)t} \prod_k S_{mn,k} \equiv [\rho_{mn}(t)]_{g_k=0} \prod_k S_{mn,k} \quad (3.6)$$

where we used (2.9). Omitting the mode index  $k$  for simplicity, the expression for  $S_{mn}$  for each mode in the product is

$$S_{mn} = Z^{-1} \text{Tr}[e^{-it\gamma_m} e^{-\beta\omega a^\dagger a} e^{it\gamma_n}] \quad (3.7)$$

where the trace is in the space of that mode, and we defined

$$\gamma_m = \omega a^\dagger a + \lambda_m(g^* a + g a^\dagger) \quad (3.8)$$

The partition function  $Z$  is given in (2.7). Relations (3.6)–(3.8) already illustrate one of our main results: apart from the phase factor which would be present in the noninteracting case anyway, the system energy eigenvalues  $E_n$  do not enter in the expression for  $\rho_{mn}(t)$ . The interesting time dependence is controlled by the eigenvalues  $\lambda_n$  of the “pointer observable” operator  $A_S$  (and by the heat-bath coupling parameters  $\omega_k$  and  $g_k$ ).

In order to evaluate the trace in (3.7), we use the coherent-state approach. We have

$$Z S_{mn} = \int d^2 z_0 d^2 z_1 d^2 z_2 \langle z_0 | e^{-it\gamma_m} | z_1 \rangle \langle z_1 | e^{-\beta\omega a^\dagger a} | z_2 \rangle \langle z_2 | e^{it\gamma_n} | z_0 \rangle \quad (3.9)$$

The normal-ordering formula (3.5) then yields for the middle term,

$$\begin{aligned} \langle z_1 | e^{-\beta\omega a^\dagger a} | z_2 \rangle &= \langle z_1 | z_2 \rangle e^{z_1^*(e^{-\beta\omega} - 1) z_2} \\ &= \exp[z_1^* z_2 - \frac{1}{2} |z_1|^2 - \frac{1}{2} |z_2|^2 + z_1^*(e^{-\beta\omega} - 1) z_2] \end{aligned} \quad (3.10)$$

In order to evaluate the first and last factors in (3.9) we define shifted operators

$$\eta = a + \lambda_m \omega^{-1} g \quad (3.11)$$

in terms of which we have

$$\gamma_m = \omega \eta^\dagger \eta - \lambda_m^2 \omega^{-1} |g|^2 \quad (3.12)$$

Since  $\eta$  and  $\eta^\dagger$  still satisfy the bosonic commutation relation  $[\eta, \eta^\dagger] = 1$ , the normal-ordering formula applies. Thus, for the first factor in (3.9), for instance, we get

$$\langle z_0 | e^{-it\gamma_m} | z_1 \rangle = e^{it\lambda_m^2(|g|^2/\omega)} \langle z_0 | z_1 \rangle e^{(e^{-i\omega t} - 1)(z_0^* + \lambda_m(g^*/\omega))(z_1 + \lambda_m(g/\omega))} \quad (3.13)$$

Collecting all these expressions, one concludes that the calculation of  $S_{mn}$  involves six Gaussian integrations over the real and imaginary parts of the variables  $z_0, z_1, z_2$ . This is a rather lengthy calculation but it can be carried out in closed form. The result, with indices  $k$  restored, is

$$S_{mn,k} = \exp(-\omega_k^{-2} |g_k|^2 P_{mn,k}) \quad (3.14)$$

where

$$P_{mn,k} = 2(\lambda_m - \lambda_n)^2 \sin^2 \frac{\omega_k t}{2} \coth \frac{\beta\omega_k}{2} + i(\lambda_m^2 - \lambda_n^2)(\sin \omega_k t - \omega_k t) \quad (3.15)$$

The expression (3.15), with (3.14), when inserted in (3.6), is the principal result of this section. It will be discussed in the next section. Here we note that in the studies of systems involving the bosonic heat bath one frequently adds the “renormalization” term<sup>2,29</sup> in the Hamiltonian,

$$H = H_S + H_B + H_I + H_R \quad (3.16)$$

where in our case

$$H_R = A_S^2 \sum_k \omega_k^{-1} |g_k|^2 \quad (3.17)$$

The role of this renormalization has been reviewed in ref. 29. Here we only notice that the sole effect of adding this term in our calculation is to modify the imaginary part of  $P_{mn,k}$  which plays no role in our subsequent discussion. The modified expression is

$$P_{mn,k} = 2(\lambda_m - \lambda_n)^2 \sin^2 \frac{\omega_k t}{2} \coth \frac{\beta\omega_k}{2} + i(\lambda_m^2 - \lambda_n^2) \sin \omega_k t \quad (3.18)$$

#### 4. CONTINUUM LIMIT AND DISCUSSION

The results of the preceding section, (3.6), (3.14), (3.15), can be conveniently discussed if we consider magnitudes of the matrix elements of the reduced density matrix  $\rho(t)$ . We have

$$|\rho_{mn}(t)| = |\rho_{mn}(0)| \exp[-\frac{1}{4}(\lambda_m - \lambda_n)^2 \Gamma(t)] \quad (4.1)$$

where we introduced the factor  $\frac{1}{4}$  to have the expression identical to that obtained in ref. 14:

$$\Gamma(t) = 8 \sum_k \omega_k^{-2} |g_k|^2 \sin^2 \frac{\omega_k t}{2} \coth \frac{\beta \omega_k}{2} \quad (4.2)$$

These results suggest several interesting conclusions. First, the decoherence is clearly controlled by the interaction with the heat bath rather than by the system's Hamiltonian. The eigenvalues of the "pointer observable"  $A_S$  determine the rate of decoherence, while the type of the bath and coupling controls the form of the function  $\Gamma(t)$ . It is interesting to note that states with equal eigenvalues  $\lambda_m$  will remain entangled even if their energies  $E_m$  are different. As expected, the magnitude of the diagonal matrix elements remains unchanged.

Secondly, we note that  $\Gamma(t)$  is a sum of positive terms. However, for true decoherence, i.e., in order for this sum to diverge for large times, one needs a *continuum* of frequencies and interactions with the bath modes that are strong enough *at low frequencies*; see below. From this point on, our discussion of the function  $\Gamma(t)$  is basically identical to that in ref. 14 (see also ref. 1); we only outline the main points. In the continuum limit, exemplified for instance by phonon modes in solid state, we introduce the density of states  $G(\omega)$  and sum over frequencies rather than modes characterized by their wave vectors. The latter change of the integration variable introduces the factor which we will loosely write as  $dk/d\omega$ ; it must be calculated from the dispersion relation of the bosonic modes. Thus we have

$$\Gamma(t) \propto \int d\omega \frac{dk}{d\omega} G(\omega) |g(\omega)|^2 \omega^{-2} \sin^2 \frac{\omega t}{2} \coth \frac{\beta \omega}{2} \quad (4.3)$$

In ref. 14, the following choice was considered, motivated by properties of the phonon field in solids; see also refs. 8–11, 12–18, 29:

$$\frac{dk}{d\omega} G(\omega) |g(\omega)|^2 \propto \omega^n e^{-\omega/\omega_c} \quad (4.4)$$

This combination of the coupling constants and frequencies has been termed the spectral function. Here  $\omega_c$  is the Debye cutoff frequency.

Specifically, the authors of ref. 14 have analyzed the cases  $n=1$  and  $n=3$ . For  $n=1$ , three regimes were identified, defined by the time scale for thermal decoherence,  $\beta$ , which is large for low temperatures, see (1.2), and the time scale for quantum-fluctuation effects,  $\omega_c^{-1}$ . Recall that we use the units  $\hbar=1$ . The present treatment only makes sense provided  $\omega_c^{-1} \ll \beta$ .

According to ref. 14, the first, “quiet” regime  $t \ll \omega_c^{-1}$  corresponds to no significant decoherence and  $\Gamma \propto (\omega_c t)^2$ . The next, “quantum” regime,  $\omega_c^{-1} \ll t \ll \beta$ , corresponds to decoherence driven by quantum fluctuations and  $\Gamma \propto \ln(\omega_c t)$ . Finally, for  $t \gg \beta$ , in the “thermal” regime, thermal fluctuations play major role in decoherence and  $\Gamma \propto t/\beta$ .

For  $n = 3$ , decoherence is incomplete.<sup>(14)</sup> Indeed, while  $n$  must be positive for the integral in (4.3) to converge, only for  $n < 2$  we have divergent  $\Gamma(t)$  growing according to a power law for large times (in fact,  $\propto t^{2-n}$ ) in the “thermal” regime. Thus, strong enough coupling  $|g(\omega)|$  to the low-frequency modes of the heat bath is crucial for full decoherence.

In summary, we derived exact results for the model of decoherence due to energy-conserving interactions with the bosonic heat bath. We find that the spectrum of the “pointer observable” that enters the interaction with the bath controls the rate of decoherence. The precise functional form of the time dependence is determined both by the choice of heat-bath and system-bath coupling. However, for the case studied, it is universal for all pointer observables and for all the matrix elements of the reduced density matrix.

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